Online Appendix to "Large Firms and Within Firm Occupational Reallocation"

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1 Extensions

In this appendix I discuss some extensions/modifications of the current setup:

Search frictions inside the firm: I relax the assumption that there are no frictions inside the firm. Consider instead the case where a task becomes available inside the firm at some rate and a worker needs to decide whether to switch to a new task. If his beliefs regarding his current task are low enough, he is willing to switch to a new task if one becomes available. Whether he is willing to switch is captured by some threshold level of beliefs. If however a task does not become available and his belief continues to fall, he may decide that it is not worth waiting for a new task and instead separate to unemployment. This will be determined by a different belief threshold. Both thresholds depend on the number of remaining tasks, k. Alternatively, it could be the case that the worker always prefers to remain in his current task waiting for another one, rather than quit to unemployment. In subsection 1.1 below, I solve for these thresholds and characterize optimal worker behavior when there are search frictions inside the firm.

Human capital accumulation: I also extend the model to allow for the possibility of human capital accumulation. In particular, there are now two types of workers: workers who have accumulated human capital and workers without human capital. Workers who do not have human capital, accumulate it at some rate, γ. Expected output produced by workers with human capital is now $\phi\overline{\alpha}(p)$, where $\phi > 1$, while workers without human capital produce $\overline{\alpha}(p)$ as before. In the case of a separation, a worker loses the human capital he has accumulated.

Workers behave differently depending on their human capital level. More specifically, there are now

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two threshold, $\underline{p}^{nohc}(k)$ and $\underline{p}^{hc}(k)$. The solution of the model in this case, which is substantially more complicated, as well as detailed derivations are presented in subsection 1.2 below.

On-the-job search: The model can be extended to allow for on-the-job search. In particular, assume that, following an appropriate modification of the matching function, employed workers are allowed to contact other firms at rate $\eta\lambda$, where $\eta > 0$ captures the effectiveness of employed workers' search technology relative to unemployed workers. Search is costless and unobserved by the firm.

If an employed worker meets another firm, he chooses the firm where his value is the highest when receiving the wage resulting from Nash bargaining. In subsection 1.3 below, I show that this is the equilibrium outcome of an ascending auction in which the current and poaching firms place bids to attract the worker and also discuss how the assumptions behind the auction preserve the convexity of the payoff set, thus overcoming Shimer (2006)'s critique.

An employed worker with current belief p and k tasks remaining, who contacts a firm with m tasks, switches to the new firm if and only if $W(p_0, m-1) > W(p,k)$ or $p < \underline{p}^{otj}(k,m)$, where the implicit threshold $\underline{p}^{otj}(k,m)$, depends on both k and m. An employed worker's value function now becomes

$$\begin{aligned} rW(p,k) &= w(p) + \frac{1}{2}\xi^2 p^2 (1-p)^2 W_{pp}(p,k) - \delta(W(p,k) - U) \\ &\quad + \eta\lambda \left(\sum_{m=1}^{M} q_m \max\{W(p_0, m-1), W(p,k)\} - W(p,k) \right), \end{aligned}$$

where the last term captures the discrete gain to the worker if he meets a firm where his value is higher. The value of the firm, $J(p,k)$, is similarly modified to capture its loss when a worker moves to another firm.

Following similar calculations as in the model without on-the-job search in the paper, the wage function now becomes

$$\begin{aligned} w(p) &= \beta\overline{\alpha}(p) + r(1-\beta)U \\ &\quad - \eta\lambda \sum_{m=1}^{M} q_m I\{W(p_0, m-1) > W(p,k)\}\left((1-\beta)(W(p_0, m-1) - W(p,k)) + \beta J(p,k)\right). \end{aligned} \tag{1}$$

Note that the worker's wage now has an additional term compared to the case with no on-the-job search considered in the paper. Effectively, his wage is reduced by an amount proportional to his search intensity. When the worker leaves his current firm for another firm, the separation is no longer bilaterally efficient, as there are lost rents for the incumbent firm. The worker compensates his firm by an amount equal to the weighted average of the expected worker's gains, $\sum_{m=1}^{M} q_m I\left\{W\left(p_0, m-1\right) > W\left(p,k\right)\right\}\left(W\left(p_0, m-1\right) - W\left(p,k\right)\right)$, and the firm's losses, $J\left(p,k\right)$, multiplied by his job finding probability.

Unlike the model without on-the-job search, it is not possible to solve for the decision thresholds $\underline{p}\left(k\right)$ and $\underline{p}^{otj}\left(k,m\right)$ analytically, but instead they need to be computed numerically.

Directed search and wage posting: In the main model above wages are determined by generalized Nash bargaining between the worker and the firm. Li and Tian (2013) extend the setup developed in present paper, to allow for firms to post wages (rather than bargain with the workers). In addition, workers now choose to which firm they should apply (directed search). In equilibrium, workers are indifferent across firms with different number of occupations, m. That setup as well can generate the observed size-wage premium and also a negative relationship between firm size and worker separation rates.

1.1 Search Frictions Inside the Firm Derivations

I relax the assumption that there are no frictions inside the firm. Instead a task becomes available inside the firm at rate μ. Now a worker who has $k > 0$ tasks remaining needs to decide whether to switch to a new task if one becomes available.

Consider again a worker with beliefs p and k tasks remaining. If his beliefs regarding his current task are low enough he is willing to switch to a new task if one becomes available. Denote this threshold $\widehat{p}\left(k\right)$: If a new task becomes available and $p < \widehat{p}\left(k\right)$, the worker switches, otherwise he remains in his current task.

If a task does not become available and his belief continues to fall, he may decide that it is not worth waiting for a new task and instead separate to unemployment. I denote this threshold by $\widetilde{p}\left(k\right)$. Below I also consider the case where that does not happen and for a given k, the worker always prefers to remain in his current task waiting for another one, rather than quit to unemployment.

The value of a worker with posterior p and k is now given by:

$$rW\left(p,k\right) = w\left(p\right) + \frac{1}{2}\xi^2 p^2\left(1-p\right)^2 W_{pp}\left(p,k\right) - \delta\left(W\left(p,k\right) - U\right) + \max\left\{\mu\left(W\left(p_0, k-1\right) - W\left(p,k\right)\right), 0\right\}$$

Similarly the value of the firm employing him is now:

$$rJ(p,k) = \overline{\alpha}(p) - w(p) + \frac{1}{2}\xi^2 p^2 (1-p)^2 J_{pp}(p,k) - \delta J(p,k) + \max\{\mu(J(p_0, k-1) - J(p,k)), 0\}$$

Following the same steps as in Appendix C leads to the same wage expression as in the baseline model:

$$w(p) = \beta\overline{\alpha}(p) + (1-\beta) rU$$

I next solve for $\widehat{p}(k)$ and $\widetilde{p}(k)$.

I first consider the case where the worker would not accept a position in another occupation inside the firm, i.e. $p \geq \widehat{p}(k)$. Then as in Appendix D of the paper, the value of the surplus is:

$$S^{noacc}(p,k) = \frac{\overline{\alpha}(p) - rU}{r+\delta} + K_3^k p^{\frac{1}{2}-\frac{1}{2}\theta} (1-p)^{\frac{1}{2}+\frac{1}{2}\theta} \tag{2}$$

where $\theta = \sqrt{\frac{8(r+\delta)}{\xi^2} + 1}$ and K_3^k is an undetermined coefficient that depends on k.

When the worker is willing to switch occupations inside the firm, i.e. $p < \widehat{p}(k)$, straightforward calculations imply that that firm surplus is given by:

$$(r+\delta+\mu) S^{acc}(p,k) = \overline{\alpha}(p) + \mu S^{noacc}(p_0, k-1) - rU + \frac{1}{2}\xi^2 p^2 (1-p)^2 S^{acc}_{pp}(p,k)$$

The general solution to the above differential equation is given by:

$$S^{acc}(p,k) = \frac{\overline{\alpha}(p) + \mu S^{noacc}(p_0, k-1) - rU}{r+\delta+\mu} + K_1^k p^{\frac{1}{2}-\frac{1}{2}\eta} (1-p)^{\frac{1}{2}+\frac{1}{2}\eta} + K_2^k p^{\frac{1}{2}+\frac{1}{2}\eta} (1-p)^{\frac{1}{2}-\frac{1}{2}\eta} \tag{3}$$

where $\eta = \sqrt{\frac{8(r+\delta+\mu)}{\xi^2} + 1}$ and K_1^k and K_2^k are undetermined coefficients that depend on k.

In order to pin down the three undetermined coefficients, K_1^k, K_2^k and K_3^k and the two triggers, $\widehat{p}(k)$ and $\widetilde{p}(k)$, I use the following five conditions:

$$S^{acc}(\widetilde{p}(k), k) = 0$$

$$S^{acc}_p(\widetilde{p}(k), k) = 0$$

$$S^{noacc}\left(\widehat{p}\left(k\right)+,k\right)=S^{noacc}\left(p_{0},k-1\right) \tag{4}$$

$$S^{acc}\left(\widehat{p}\left(k\right)-,k\right)=S^{noacc}\left(p_{0},k-1\right) \tag{5}$$

$$S_{p}^{acc}\left(\widehat{p}\left(k\right)-,k\right)=S_{p}^{noacc}\left(\widehat{p}\left(k\right)+,k\right) \tag{6}$$

The first two equations are the value matching and smooth pasting conditions at $\widetilde{p}(k)$. The third condition implies continuity from the right for the value of the surplus at the point where the worker is now willing to accept another occupation, $\widehat{p}(k)$ (value matching). Similarly the fourth condition implies continuity from the left for the value of the surplus at $\widehat{p}(k)$. The last condition is a smooth pasting condition for beginning to accept another position at $\widehat{p}(k)$.[1] Straightforward calculations reduce the above system of five equations and five unknowns, to two equations that implicitly define $\widehat{p}(k)$ and $\widetilde{p}(k)$.

Finally, I need to consider the case where it is not optimal for the worker to quit to unemployment, but instead prefers to remain in his current task waiting for another one, no matter how low his posterior. In this case, the above system of two equations and two unknowns does not have a real solution. However I still need to solve for $\widehat{p}(k)$.

In that case, the value of the surplus when the worker does not accept another position ($p\geq\widehat{p}(k)$) is given by equation (2). When the worker is willing to accept another position ($p<\widehat{p}(k)$), the value of the surplus is given by (3). Now however, when $p\to 0$, $\lim_{p\to 0}K_{1}^{k}p^{\frac{1}{2}-\frac{1}{2}\eta}\left(1-p\right)^{\frac{1}{2}+\frac{1}{2}\eta}=K_{1}^{k}\cdot\lim_{p\to 0}p^{\frac{1}{2}-\frac{1}{2}\eta}\cdot 1=+\infty$ which follows from $\eta>1$. Since the present discount sum of the surplus produced by a bad match is given by $\frac{\alpha^{B}+\mu S^{noacc}(p_{0},k-1)-rU}{r+\delta+\mu}<\infty$, it must be the case that $K_{1}^{k}=0$, $\forall k$.

I am left with three unknowns, K_{2}^{k}, K_{3}^{k} and $\widehat{p}(k)$, which can now be solved using equations (4), (5) and (6). Straightforward calculations reduces the three by three system to one equation that implicitly defines $\widehat{p}(k)$.

1.2 Human Capital Derivations

I extend the model to allow for the possibility of human capital accumulation. There are now two types of workers: workers who have accumulated human capital and workers without human capital. Workers who do not have human capital accumulate it at some rate, γ. Expected output produced by workers

[1] See also condition (iii) in page 509 of Moscarini (2005).

with human capital is now $\phi\overline{\alpha}(p)$, where $\phi > 1$, while workers without human capital produce $\overline{\alpha}(p)$ as before. In the case of a separation, a worker loses the human capital he has accumulated.

The value of unemployment now becomes:

$$rU = b + \lambda \left(\sum_{m=1}^{M} q_m E_{p_0} W^{nohc}(p_0, m-1) \right) - \lambda U$$

where W^{nohc} is the value function for a worker without any human capital.

I start with the case of the worker that has accumulated human capital.

As in the model without human capital, it is the case that

$$\beta J^{hc}(p,k) = (1-\beta)\left(W^{hc}(p,k) - U\right), \tag{7}$$

and the wage is given by

$$w^{hc}(p) = \beta\phi\overline{\alpha}(p) + (1-\beta) rU,$$

Moreover the relevant value matching and the smooth pasting conditions are:

$$W^{hc}\left(\underline{p}^{hc}(k), k\right) = E_{p_0} W^{hc}(p_0, k-1) \tag{8}$$

and

$$W_p^{hc}\left(\underline{p}^{hc}(k), k\right) = 0 \tag{9}$$

Note that I allow the threshold, $\underline{p}^{hc}(k)$, to depend on the number of remaining tasks in the current firm and also on whether the worker has accumulated human capital.

A worker who has exhausted all available tasks in his current firm, i.e. $k = 0$, optimally quits to unemployment when:

$$W^{hc}\left(\underline{p}^{hc}(0), 0\right) = U \tag{10}$$

and

$$W_p^{hc}\left(\underline{p}^{hc}(0), 0\right) = 0 \tag{11}$$

The surplus of the match between the firm and the worker who has accumulated human capital,

$S^{hc}(p,k)$, is given by:

$$S^{hc}(p,k) = W^{hc}(p,k) + J^{hc}(p,k) - U$$

Substituting in for $W^{hc}(p,k)$ and $J^{hc}(p,k)$ leads to:

$$(r+\delta) S^{hc}(p,k) = \phi\overline{\alpha}(p) - rU + \frac{1}{2}\zeta_{hc}^2 p^2 (1-p)^2 S_{pp}^{hc}(p,k)$$

where

$$\zeta_{hc} = \frac{\phi\left(\alpha^G - \alpha^B\right)}{\sigma}$$

The general solution to the above differential equation is given by:

$$S^{hc}(p,k) = \frac{\phi\overline{\alpha}(p) - rU}{r+\delta} + K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} + K_2^k p^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}$$

where

$$\theta_{hc} = \sqrt{\frac{8(r+\delta)}{\zeta_{hc}^2} + 1}$$

and K_1^k and K_2^k are undetermined coefficients that depend on k. When $p \to 1$ however,

$$\lim_{p\to 1} K_2^k p^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} = K_2^k \cdot 1 \cdot \lim_{p\to 1} (1-p)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} = +\infty,$$

which follows from $\theta_{hc} > 1$. Since the present discount sum of the output produced by a good match is given by $\frac{\phi\alpha^G}{r+\delta} < \infty$, it must be the case that $K_2^k = 0$, $\forall k$. Thus:

$$S^{hc}(p,k) = \frac{\phi\overline{\alpha}(p) - rU}{r+\delta} + K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}$$

where K_1^k is an undetermined coefficient. Moreover:

$$S_p^{hc}(p,k) = \frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta} + K_1^k \left(\frac{1}{2} - \frac{1}{2}\theta_{hc} - p\right) p^{-\frac{1}{2}-\frac{1}{2}\theta_{hc}} (1-p)^{-\frac{1}{2}+\frac{1}{2}\theta_{hc}}$$

Consider the case where $k = 0$. Using equation (7), I can rewrite the value matching and smooth pasting conditions (equations (10) and (11) respectively), in terms of $S(\cdot)$ and use them to pin down $\underline{p}(0)$ and K_1^0.

From equation (11), for the case of $k=0$, I obtain:

$$S_p^{hc}\left(\underline{p}^{hc}(0),0\right)=0$$

$$\frac{\phi\left(\alpha^G-\alpha^B\right)}{r+\delta}+K_1^0\left(\frac{1}{2}-\frac{1}{2}\theta_{hc}-\underline{p}^{hc}(0)\right)\underline{p}^{hc}(0)^{-\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(0)\right)^{-\frac{1}{2}+\frac{1}{2}\theta_{hc}}=0$$

$$K_1^0=-\frac{\phi\left(\alpha^G-\alpha^B\right)}{r+\delta}\left(\frac{1}{2}-\frac{1}{2}\theta_{hc}-\underline{p}^{hc}(0)\right)^{-1}\underline{p}^{hc}(0)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(0)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}$$

Similarly there's a smooth pasting condition for $k>0$, that implies that

$$S_p^{hc}\left(\underline{p}^{hc}(k),k\right)=0$$

$$K_1^k=-\frac{\phi\left(\alpha^G-\alpha^B\right)}{r+\delta}\left(\frac{1}{2}-\frac{1}{2}\theta_{hc}-\underline{p}^{hc}(k)\right)^{-1}\underline{p}^{hc}(k)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} \tag{12}$$

Using equation (10) and substituting for K_1^0 I obtain

$$W^{hc}\left(\underline{p}^{hc}(0),0\right)=U$$

$$S^{hc}\left(\underline{p}^{hc}(0),0\right)=0$$

$$\frac{\phi\overline{\alpha}\left(\underline{p}^{hc}(0)\right)-rU}{r+\delta}+K_1^0\underline{p}^{hc}(0)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(0)\right)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}=0$$

$$\begin{aligned}&\frac{\phi\overline{\alpha}\left(\underline{p}^{hc}(0)\right)-rU}{r+\delta}\\&-\frac{\phi\left(\alpha^G-\alpha^B\right)}{r+\delta}\left(\frac{1}{2}-\frac{1}{2}\theta_{hc}-\underline{p}(0)\right)^{-1}\underline{p}^{hc}(0)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(0)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\times\\&\underline{p}^{hc}(0)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(0)\right)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\\=\;&0\end{aligned}$$

which straightforward calculations lead to

$$\underline{p}^{hc}(0)=\frac{\left(\theta_{hc}-1\right)\left(rU-\phi\alpha^B\right)}{\left(\theta_{hc}-1\right)\left(rU-\phi\alpha^B\right)+\left(\theta_{hc}+1\right)\left(\phi\alpha^G-rU\right)}.$$

In the case of where $k > 0$, equation (9) leads to:

$$S_p^{hc}\left(\underline{p}^{hc}(k), k\right) = 0$$

$$\frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta} + K_1^k\left(\frac{1}{2} - \frac{1}{2}\theta_{hchc} - \underline{p}^{hc}(k)\right)\underline{p}^{hc}(k)^{-\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{-\frac{1}{2}+\frac{1}{2}\theta_{hc}} = 0$$

$$K_1^k = -\frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta}\left(\frac{1}{2} - \frac{1}{2}\theta_{hc} - \underline{p}^{hc}(k)\right)^{-1}\underline{p}^{hc}(k)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}$$

As before, I'm going to use the value matching condition (8) for the surplus function and substitute in for K_1^k (12). I have that

$$W^{hc}\left(\underline{p}^{hc}(k), k\right) = E_{p_0}W^{hc}(p_0, k-1),$$

but since

$$\beta S\left(\underline{p}^{hc}(k), k\right) + U = W^{hc}\left(\underline{p}^{hc}(k), k\right),$$

I can write

$$\beta S\left(\underline{p}^{hc}(k), k\right) = E_{p_0}W^{hc}(p_0, k-1) - U.$$

$$\beta\frac{\phi\overline{\alpha}\left(\underline{p}^{hc}(k)\right) - rU}{r+\delta} + \beta K_1^k\underline{p}^{hc}(k)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} = E_{p_0}W^{hc}(p_0, k-1) - U$$

$$\begin{aligned}
&\beta\frac{\phi\overline{\alpha}\left(\underline{p}^{hc}(k)\right) - rU}{r+\delta} \\
&+\beta\left(-\frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta}\left(\frac{1}{2} - \frac{1}{2}\theta_{hc} - \underline{p}^{hc}(k)\right)^{-1}\underline{p}^{hc}(k)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\right)\times \\
&\underline{p}^{hc}(k)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}\left(1-\underline{p}^{hc}(k)\right)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} \\
= \;& E_{p_0}W^{hc}(p_0, k-1) - U
\end{aligned}$$

which leads to

$$\underline{p}^{hc}(k) = \frac{(\theta_{hc} - 1)\left((r+\delta)E_{p_0}W^{hc}(p_0, k-1) - (r+\delta-r\beta)U - \beta\phi\alpha^B\right)}{-2(r+\delta)E_{p_0}W^{hc}(p_0, k-1) + 2(r+\delta-r\beta)U + \beta\theta_{hc}\phi\left(\alpha^G - \alpha^B\right) + \beta\phi\alpha^G + \beta\phi\alpha^B} \tag{13}$$

Note that straightforward calculations show that $S_{pp}^{hc}(p,k) > 0$, and therefore $W_{pp}^{hc}(p,k) > 0$, for all

p and k. Given that $W_p^{hc}\left(\underline{p}^{hc}(k), k\right) = 0$, this implies that $W_p^{hc}(p,k) > 0$ for all $p > \underline{p}^{hc}(k)$.

Summarizing:

$$\underline{p}^{hc}(0) = \frac{(\theta_{hc}-1)\left(rU - \phi\alpha^B\right)}{(\theta_{hc}-1)\left(rU - \phi\alpha^B\right) + (\theta_{hc}+1)\left(\phi\alpha^G - rU\right)}$$

$$\underline{p}^{hc}(k) = \frac{(\theta_{hc}-1)\left((r+\delta)E_{p_0}W^{hc}(p_0,k-1) - (r+\delta-r\beta)U - \beta\phi\alpha^B\right)}{-2(r+\delta)E_{p_0}W^{hc}(p_0,k-1) + 2(r+\delta-r\beta)U + \beta\theta_{hc}\phi\left(\alpha^G-\alpha^B\right) + \beta\phi\alpha^G + \beta\phi\alpha^B}$$

I next solve the case for the worker who hasn't accumulated human capital. As before, surplus sharing implies that

$$\beta J^{nohc}(p,k) = (1-\beta)\left(W^{nohc}(p,k) - U\right).$$

I first solve for the wage.

The value of the firm employing a worker without human capital is given by:

$$rJ^{nohc}(p,k) = \overline{\alpha}(p) - w^{nohc}(p) + \frac{1}{2}\zeta_{nohc}^2 p^2(1-p)^2 J_{pp}^{nohc}(p,k) + \gamma\left(J^{hc}(p,k) - J^{nohc}(p,k)\right) - \delta J^{nohc}(p,k),$$

where

$$\zeta_{nohc} = \frac{\left(\alpha^G - \alpha^B\right)}{\sigma}.$$

The value of the worker is given by:

$$\begin{aligned} r\,W^{nohc}(p,k) &= w^{nohc}(p) + \frac{1}{2}\zeta_{nohc}^2 p^2(1-p)^2 W_{pp}^{nohc}(p,k) \\ &\quad + \gamma\left(W^{hc}(p,k) - W^{nohc}(p,k)\right) - \delta\left(W^{nohc}(p,k) - U\right) \end{aligned}$$

Multiplying the above equation by $1-\beta$ and subtracting $(1-\beta)(r+\gamma)U$ leads to:

$$\begin{aligned} & (1-\beta)(r+\gamma)\left(W^{nohc}(p,k) - U\right) \\ = \; & (1-\beta)w^{nohc}(p) + \frac{1}{2}(1-\beta)\zeta_{nohc}^2 p^2(1-p)^2 W_{pp}^{nohc}(p,k) + \gamma(1-\beta)W^{hc}(p,k) \\ & -\delta(1-\beta)\left(W^{nohc}(p,k) - U\right) - (1-\beta)(r+\gamma)U \end{aligned}$$

Multiplying the value of the firm equation by β and subtracting the above equation and using the surplus sharing condition leads to:

$$
\begin{aligned}
\beta\left(r+\gamma\right)J^{nohc}\left(p,k\right)-\left(1-\beta\right)\left(r+\gamma\right)\Big(W^{nohc}\left(p,k\right)-U\Big) &= \\
\beta\overline{\alpha}\left(p\right)-\beta w^{nohc}\left(p\right)-\left(1-\beta\right)w^{nohc}\left(p\right)+\frac{1}{2}\beta\zeta_{nohc}^{2}p^{2}\left(1-p\right)^{2}J_{pp}^{nohc}\left(p,k\right) \\
-\frac{1}{2}\left(1-\beta\right)\zeta_{nohc}^{2}p^{2}\left(1-p\right)^{2}W_{pp}^{nohc}\left(p,k\right)+\beta\gamma J^{hc}\left(p,k\right) \\
-\gamma\left(1-\beta\right)W^{hc}\left(p,k\right)-\delta\beta J^{nohc}\left(p,k\right)+\delta\left(1-\beta\right)\Big(W^{nohc}\left(p,k\right)-U\Big) \\
+\left(1-\beta\right)\left(r+\gamma\right)U
\end{aligned}
$$

$$
\begin{aligned}
0 &= \overline{\alpha}\left(p\right)-w^{nohc}\left(p\right)+\frac{1}{2}\zeta_{nohc}^{2}p^{2}\left(1-p\right)^{2}\Big(\beta J_{pp}^{nohc}\left(p,k\right)-\left(1-\beta\right)W_{pp}^{nohc}\left(p,k\right)\Big) \\
&+\beta\gamma J^{hc}\left(p,k\right)-\gamma\left(1-\beta\right)W^{hc}\left(p,k\right)+\left(1-\beta\right)\left(r+\gamma\right)U
\end{aligned}
$$

since

$$\beta J^{nohc}\left(p,k\right)-\left(1-\beta\right)\Big(W^{nohc}\left(p,k\right)-U\Big)=0.$$

Note that taking derivative with respect to p twice leads to:

$$\beta J_{pp}^{nohc}\left(p,k\right)=\left(1-\beta\right)W_{pp}^{nohc}\left(p,k\right)$$

So I have:

$$
\begin{aligned}
w^{nohc}\left(p\right) &= \beta\overline{\alpha}\left(p\right)+\left(1-\beta\right)\left(r+\gamma\right)U \\
&\quad+\beta\gamma J^{hc}\left(p,k\right)-\gamma\left(1-\beta\right)W^{hc}\left(p,k\right)
\end{aligned}
$$

But:

$$\beta J^{hc}\left(p,k\right)-\left(1-\beta\right)\Big(W^{hc}\left(p,k\right)-U\Big)=0$$

$$\beta\gamma J^{hc}\left(p,k\right)-\left(1-\beta\right)\gamma W^{hc}\left(p,k\right)=-\left(1-\beta\right)\gamma U$$

Thus the wage equation becomes:

$$w^{nohc}\left(p\right)=\beta\overline{\alpha}\left(p\right)+\left(1-\beta\right)\left(r+\gamma\right)U-\left(1-\beta\right)\gamma U$$

$$w^{nohc}(p) = \beta\overline{\alpha}(p) + (1-\beta)\, rU$$

I next solve for the threshold, $\underline{p}^{nohc}(k)$.

The relevant value matching and the smooth pasting conditions are now:

$$W^{nohc}\left(\underline{p}^{nohc}(k), k\right) = E_{p_0} W^{nohc}(p_0, k-1)$$

and

$$W_p^{nohc}\left(\underline{p}^{nohc}(k), k\right) = 0$$

Note that I allow the threshold, $\underline{p}^{nohc}(k)$, to depend on the number of remaining tasks in the current firm and also on the fact that the worker has not accumulated human capital.

A worker who has exhausted all available tasks in his current firm, i.e. $k = 0$, optimally quits to unemployment when:

$$W^{nohc}\left(\underline{p}^{nohc}(0), 0\right) = U \tag{14}$$

and

$$W_p^{nohc}\left(\underline{p}^{nohc}(0), 0\right) = 0 \tag{15}$$

The surplus of the match between the firm and the worker who has not accumulated human capital, $S^{nohc}(p,k)$, is given by

$$S^{nohc}(p,k) = W^{nohc}(p,k) + J^{nohc}(p,k) - U$$

$$(r+\gamma)\, S^{nohc}(p,k) = (r+\gamma)\, W^{nohc}(p,k) + (r+\gamma)\, J^{nohc}(p,k) - (r+\gamma)\, U.$$

Substituting in for $W^{nohc}(p,k)$ and $J^{nohc}(p,k)$ leads to

$$\begin{aligned}(r+\delta+\gamma)\, S^{nohc}(p,k) &= \overline{\alpha}(p) - rU + \frac{1}{2}\zeta_{nohc}^2 p^2 (1-p)^2 S_{pp}^{nohc}(p,k) \\ &\quad + \gamma S^{hc}(p,k).\end{aligned}$$

But I know from above that

$$S^{hc}(p,k) = \frac{\phi\overline{\alpha}(p) - rU}{r+\delta} + K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}.$$

Substituting for $S^{hc}(p,k)$ leads to

$$\begin{aligned}(r+\delta+\gamma) S^{nohc}(p,k) &= p\left(\alpha^G - \alpha^B\right) + \alpha^B - rU + \frac{1}{2}\zeta_{nohc}^2 p^2 (1-p)^2 S_{pp}^{nohc}(p,k) \\ &\quad + \gamma \frac{\phi\left(p\left(\alpha^G-\alpha^B\right)+\alpha^B\right) - rU}{r+\delta} + \gamma K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}} (1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{1}{r+\delta+\gamma}\zeta_{nohc}^2 p^2 (1-p)^2 S_{pp}^{nohc}(p,k) \\ = \; & S^{nohc}(p,k) \\ & -\frac{\alpha^G-\alpha^B}{r+\delta}\frac{r+\delta+\gamma\phi}{r+\delta+\gamma}p - \frac{\gamma}{r+\delta+\gamma}K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}(1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}} \\ & -\frac{1}{r+\delta+\gamma}\left(\alpha^B - rU + \frac{\gamma\left(\phi\alpha^B - rU\right)}{r+\delta}\right),\end{aligned}$$

subject to when $k=0$

$$S^{nohc}\left(\underline{p}^{nohc}(k), k\right) = 0$$

$$S_p^{nohc}\left(\underline{p}^{nohc}(k), k\right) = 0,$$

and when $k>0$

$$S^{nohc}\left(\underline{p}^{nohc}(k), k\right) = S^{nohc}(p_0, k-1)$$

$$S_p^{nohc}\left(\underline{p}^{nohc}(k), k\right) = 0.$$

I rewrite the above differential equation as follows

$$k_3 t^2 (1-t)^2 \ddot{y}(t) = y(t) + k_1 t + g(W) t^{\eta} (1-t)^{1-\eta} + k_2,$$

where

$$y(t) = S^{nohc}(p,k),$$

$$k_3 = \frac{1}{2}\frac{1}{r+\delta+\gamma}\zeta_{nohc}^2,$$

$$k_1 = -\frac{\alpha^G - \alpha^B}{r+\delta}\frac{r+\delta+\gamma\phi}{r+\delta+\gamma},$$

$$g(W) = -\frac{\gamma}{r+\delta+\gamma}K_1^k,$$

$$K_1^0 = -\frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta}\left(\frac{1}{2} - \frac{1}{2}\theta_{hc} - \underline{p}^{hc}(0)\right)^{-1}\underline{p}^{hc}(0)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1 - \underline{p}^{hc}(0)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}},$$

$$K_1^k = -\frac{\phi\left(\alpha^G - \alpha^B\right)}{r+\delta}\left(\frac{1}{2} - \frac{1}{2}\theta_{hc} - \underline{p}^{hc}(k)\right)^{-1}\underline{p}^{hc}(k)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}}\left(1 - \underline{p}^{hc}(k)\right)^{\frac{1}{2}-\frac{1}{2}\theta_{hc}},$$

$$\underline{p}^{hc}(0) = \frac{(\theta_{hc}-1)\left(rU - \phi\alpha^B\right)}{(\theta_{hc}-1)\left(rU - \phi\alpha^B\right) + (\theta_{hc}+1)\left(\phi\alpha^G - rU\right)},$$

$$\underline{p}^{hc}(k) = \frac{(\theta_{hc}-1)\left((r+\delta)E_{p_0}W^{hc}(p_0, k-1) - (r+\delta-r\beta)U - \beta\phi\alpha^B\right)}{-2(r+\delta)E_{p_0}W^{hc}(p_0, k-1) + 2(r+\delta-r\beta)U + \beta\theta_{hc}\phi\left(\alpha^G - \alpha^B\right) + \beta\phi\alpha^G + \beta\phi\alpha^B},$$

$$\theta_{hc} = \sqrt{\frac{8(r+\delta)}{\zeta_{hc}^2} + 1},$$

$$\eta = \frac{1}{2} - \frac{1}{2}\theta_{hc},$$

$$k_2 = -\frac{1}{r+\delta+\gamma}\left(\alpha^B - rU + \frac{\gamma\left(\phi\alpha^B - rU\right)}{r+\delta}\right),$$

subject to

$$y(t_0) = W,$$

$$\dot{y}(t_0) = 0$$

where $y(t_0) = S_p^{nohc}\left(\underline{p}^{nohc}(k), k\right)$ and W is equal to either 0 (when $k = 0$) or $E_{p_0}S^{nohc}(p_0, k-1)$ (when $k > 0$).

In order to solve the equation, I follow Brockett (1970), Chapter 1 (see also online appendix of Papageorgiou (2017)).

Let

$$\begin{aligned} x_1(t) &= y(t), \\ x_2(t) &= \dot{y}(t). \end{aligned}$$

Then

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= \ddot{x}(t) = \frac{1}{k_3 t^2 (1-t)^2} x_1(t) + Z(t).
\end{aligned}$$

where

$$Z(t) = \frac{1}{k_3 t^2 (1-t)^2} \left[k_1 t + g(W) t^{\eta} (1-t)^{1-\eta} + k_2 \right].$$

Moreover let

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}'.$$

Then

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{1}{k_3 t^2 (1-t)^2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} Z(t).$$

I now use the "Variation of Constants formula". According to Brockett (1970), Chapter 1.6, if $\Phi(t, t_0)$ is the transition matrix for $\dot{x}(t) = A(t) x(t)$, then the unique solution of $\dot{x}(t) = A(t) x(t) + f(t)$; $x(t_0) = x_0$ is given by:

$$x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} Z(\tau) \, d\tau,$$

where $\Phi(t, t_o)$ is the transition matrix. The transition matrix is defined as the solution Φ of the matrix differential equation Brockett (1970), Chapter 1.3)

$$\begin{aligned}
\frac{d}{dt} \Phi(t, t_o) &= A(t) \Phi(t, t_o) \\
\Phi(t_0, t_o) &= I,
\end{aligned}$$

where I is the identity matrix.

For the problem at hand, I have:

$$\dot{x}(t) = A(t) x(t)$$

or

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{1}{k_3 t^2 (1-t)^2} & 0 \end{bmatrix} x(t).$$

Moreover let

$$\begin{aligned} x_0 &= \begin{bmatrix} x_{10} & x_{20} \end{bmatrix}' \\ x_{10} &= y(t_0) \\ x_{20} &= \dot{y}(t_0). \end{aligned}$$

The solution to the homogeneous differential equation

$$\ddot{y}(t) = \frac{1}{k_3 t^2 (1-t)^2} y(t),$$

is given by

$$y(t) = c_1 t^q (1-t)^{1-q} + c_2 t^{1-q} (1-t)^q,$$

where $q = \frac{1}{2}\left(1 - \sqrt{\frac{4+k_3}{k_3}}\right)$ and c_1 and c_2 are undetermined coefficients.

Given that I know the solution to the homogeneous, I can write

$$\Phi(t, t_0) x_0 = \begin{bmatrix} t^q (1-t)^{1-q} & t^{1-q} (1-t)^q \\ \frac{d}{dt}\left(t^q (1-t)^{1-q}\right) & \frac{d}{dt}\left(t^{1-q} (1-t)^q\right) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Phi(t, t_0) x_0 = \begin{bmatrix} t^q (1-t)^{1-q} & t^{1-q} (1-t)^q \\ t^{q-1} (1-t)^{-q} (q-t) & t^{-q} (1-t)^{q-1} (1-q-t) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where c_1 and c_2 are undetermined coefficients.

I next solve for c_1 and c_2 as a function of x_0.

At $t = t_0$, since $\Phi(t_0, t_0) = I$, I have

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} t_0^q (1-t_0)^{1-q} & t_0^{1-q} (1-t_0)^q \\ t_0^{q-1} (1-t_0)^{-q} (q-t_0) & t_0^{-q} (1-t_0)^{q-1} (1-q-t_0) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1-2q} \begin{bmatrix} t_0^{-q}(1-t_0)^{q-1}(1-q-t_0) & -t_0^{1-q}(1-t_0)^q \\ -t_0^{q-1}(1-t_0)^{-q}(q-t_0) & t_0^q(1-t_0)^{1-q} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

Therefore, substituting in for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, in the equation I had leads to

$$\Phi(t,t_0)x_0 = \begin{bmatrix} t^q(1-t)^{1-q} & t^{1-q}(1-t)^q \\ t^{q-1}(1-t)^{-q}(q-t) & t^{-q}(1-t)^{q-1}(1-q-t) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{aligned} \Phi(t,t_0)x_0 &= \begin{bmatrix} t^q(1-t)^{1-q} & t^{1-q}(1-t)^q \\ t^{q-1}(1-t)^{-q}(q-t) & t^{-q}(1-t)^{q-1}(1-q-t) \end{bmatrix} \\ & \frac{1}{1-2q} \begin{bmatrix} t_0^{-q}(1-t_0)^{q-1}(1-q-t_0) & -t_0^{1-q}(1-t_0)^q \\ -t_0^{q-1}(1-t_0)^{-q}(q-t_0) & t_0^q(1-t_0)^{1-q} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Phi(t,t_0) &= \frac{1}{1-2q} \begin{bmatrix} t^q(1-t)^{1-q} & t^{1-q}(1-t)^q \\ t^{q-1}(1-t)^{-q}(q-t) & t^{-q}(1-t)^{q-1}(1-q-t) \end{bmatrix} \\ & \begin{bmatrix} t_0^{-q}(1-t_0)^{q-1}(1-q-t_0) & -t_0^{1-q}(1-t_0)^q \\ -t_0^{q-1}(1-t_0)^{-q}(q-t_0) & t_0^q(1-t_0)^{1-q} \end{bmatrix}. \end{aligned}$$

In other words, the solution to the homogeneous is given by

$$x(t) = \Phi(t,t_0)x_0,$$

where $\Phi(t,t_0)$ is defined above.

I know that for $t_0 = \underline{p}+$

$$y(t_0) = W,$$

and

$$\dot{y}(t_0) = 0.$$

Changing the notation

$$
\begin{aligned}
x_0 &= \begin{bmatrix} x_{10} & x_{20} \end{bmatrix}' \\
x_{10} &= y(t_0) = W \\
x_{20} &= \dot{y}(t_0) = 0
\end{aligned}
$$

Then the solution to the homogeneous becomes

$$
x(t) = \Phi(t, t_0) \begin{bmatrix} W \\ 0 \end{bmatrix}.
$$

Using the variation of constants formula, I can now solve the original differential equation

$$
x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} Z(\tau)\, d\tau
$$

$$
\begin{aligned}
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \frac{1}{1-2q} \begin{bmatrix} t^q (1-t)^{1-q} & t^{1-q} (1-t)^q \\ t^{q-1} (1-t)^{-q} (q-t) & t^{-q} (1-t)^{q-1} (1-q-t) \end{bmatrix} \\
& \quad \begin{bmatrix} t_0^{-q} (1-t_0)^{q-1} (1-q-t_0) & -t_0^{1-q} (1-t_0)^q \\ -t_0^{q-1} (1-t_0)^{-q} (q-t_0) & t_0^q (1-t_0)^{1-q} \end{bmatrix} \begin{bmatrix} W \\ 0 \end{bmatrix} \\
& \quad + \int_{t_0}^{t} \frac{1}{1-2q} \begin{bmatrix} t^q (1-t)^{1-q} & t^{1-q} (1-t)^q \\ t^{q-1} (1-t)^{-q} (q-t) & t^{-q} (1-t)^{q-1} (1-q-t) \end{bmatrix} \cdot \\
& \quad \begin{bmatrix} \tau^{-q} (1-\tau)^{q-1} (1-q-\tau) & -\tau^{1-q} (1-\tau)^q \\ -\tau^{q-1} (1-\tau)^{-q} (q-\tau) & \tau^q (1-\tau)^{1-q} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} Z(\tau)\, d\tau
\end{aligned}
$$

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{1-2q} \cdot
$$

$$
\begin{bmatrix} t^q (1-t)^{1-q} t_0^{-q} (1-t_0)^{q-1} (1-q-t_0) W - t^{1-q} (1-t)^q t_0^{q-1} (1-t_0)^{-q} (q-t_0) W \\ t^{q-1} (1-t)^{-q} (q-t) t_0^{-q} (1-t_0)^{q-1} (1-q-t_0) W - t^{-q} (1-t)^{q-1} (1-q-t) t_0^{q-1} (1-t_0)^{-q} (q-t_0) W \end{bmatrix}
$$

$$+\frac{1}{1-2q}\begin{bmatrix} t^{q}(1-t)^{1-q} & t^{1-q}(1-t)^{q} \\ t^{q-1}(1-t)^{-q}(q-t) & t^{-q}(1-t)^{q-1}(1-q-t) \end{bmatrix}.$$

$$\begin{bmatrix} \int_{t_0}^{t}\frac{1}{k_3}\left[k_1\tau+g(W)\tau^{\eta}(1-\tau)^{1-\eta}+k_2\right]\left(-\tau^{-1-q}(1-\tau)^{q-2}\right)d\tau \\ \int_{t_0}^{t}\frac{1}{k_3}\left[k_1\tau+g(W)\tau^{\eta}(1-\tau)^{1-\eta}+k_2\right]\left(\tau^{q-2}(1-\tau)^{-1-q}\right)d\tau \end{bmatrix}$$

In other words, the solution to the differential equation is of the following form (with no undermined coefficients)

$$\begin{aligned} y(t) &= \frac{1}{1-2q}\left[t^{q}(1-t)^{1-q}t_0^{-q}(1-t_0)^{q-1}(1-q-t_0)W-t^{1-q}(1-t)^{q}t_0^{q-1}(1-t_0)^{-q}(q-t_0)W\right] \\ &+\frac{1}{k_3(1-2q)}t^{q}(1-t)^{1-q}\int_{t_0}^{t}\left[k_1\tau+g(W)\tau^{\eta}(1-\tau)^{1-\eta}+k_2\right]\left(-\tau^{-1-q}(1-\tau)^{q-2}\right)d\tau \\ &+\frac{1}{k_3(1-2q)}t^{1-q}(1-t)^{q}\int_{t_0}^{t}\left[k_1\tau+g(W)\tau^{\eta}(1-\tau)^{1-\eta}+k_2\right]\left(\tau^{q-2}(1-\tau)^{-1-q}\right)d\tau \end{aligned}$$

$$\begin{aligned} y(t) &= \frac{1}{1-2q}t^{q}(1-t)^{1-q}[t_0^{-q}(1-t_0)^{q-1}(1-q-t_0)W \\ &+\frac{1}{k_3}(-k_1\int_{t_0}^{t}\tau^{-q}(1-\tau)^{q-2}d\tau \\ &-g(W)\int_{t_0}^{t}\tau^{\eta-1-q}(1-\tau)^{-\eta+q-1}d\tau-k_2\int_{t_0}^{t}\tau^{-1-q}(1-\tau)^{q-2}d\tau)] \\ &+\frac{1}{1-2q}t^{1-q}(1-t)^{q}[t_0^{q-1}(1-t_0)^{-q}(t_0-q)W \\ &+\frac{1}{k_3}(k_1\int_{t_0}^{t}\tau^{q-1}(1-\tau)^{-1-q}d\tau \\ &+g(W)\int_{t_0}^{t}\tau^{q+\eta-2}(1-\tau)^{-\eta-q}d\tau+k_2\int_{t_0}^{t}\tau^{q-2}(1-\tau)^{-1-q}d\tau)], \end{aligned}$$

where $t_0 \equiv \underline{p}+$ and $q=\frac{1}{2}\left(1-\sqrt{\frac{4+k_3}{k_3}}\right)$.

But using integration by substitution (changing the variable in each of the integrals to $\frac{1}{\tau}$), the six integrals above are equal to:

$$\int_{t_0}^{t}\tau^{-q}(1-\tau)^{q-2}d\tau=\frac{1}{q-1}\left(\left(\frac{1}{t_0}-1\right)^{q-1}-\left(\frac{1}{t}-1\right)^{q-1}\right)$$

$$\int_{t_0}^{t}\tau^{\eta-1-q}(1-\tau)^{-\eta+q-1}d\tau=\frac{1}{q-\eta}\left(\left(\frac{1}{t_0}-1\right)^{q-\eta}-\left(\frac{1}{t}-1\right)^{q-\eta}\right)$$

$$\int_{t_0}^{t} \tau^{-1-q}(1-\tau)^{q-2}\,d\tau = \frac{1}{q(q-1)}\left(\left(\frac{1}{t_0}-1\right)^{q-1}\left(q\frac{1}{t_0}-\frac{1}{t_0}+1\right)-\left(\frac{1}{t}-1\right)^{q-1}\left(q\frac{1}{t}-\frac{1}{t}+1\right)\right)$$

$$\int_{t_0}^{t} \tau^{q-1}(1-\tau)^{-1-q}\,d\tau = \frac{1}{q}\left(\left(\frac{1}{t}-1\right)^{-q}-\left(\frac{1}{t_0}-1\right)^{-q}\right)$$

$$\int_{t_0}^{t} \tau^{q+\eta-2}(1-\tau)^{-\eta-q}\,d\tau = \frac{1}{\eta+q-1}\left(\left(\frac{1}{t}-1\right)^{1-q-\eta}-\left(\frac{1}{t_0}-1\right)^{1-q-\eta}\right)$$

$$\int_{t_0}^{t} \tau^{q-2}(1-\tau)^{-1-q}\,d\tau = \frac{1}{q(q-1)}\left(\left(\frac{1}{t}-1\right)^{-q}\left(q\frac{1}{t}-1\right)-\left(\frac{1}{t_0}-1\right)^{-q}\left(q\frac{1}{t_0}-1\right)\right).$$

So I end up with

$$\begin{aligned}
y(t) \quad = \quad & \frac{1}{1-2q}t^{q}(1-t)^{1-q}[t_0^{-q}(1-t_0)^{q-1}(1-q-t_0)W \\
& +\frac{1}{k_3}(-k_1\int_{t_0}^{t}\tau^{-q}(1-\tau)^{q-2}\,d\tau \\
& -g(W)\int_{t_0}^{t}\tau^{\eta-1-q}(1-\tau)^{-\eta+q-1}\,d\tau - k_2\int_{t_0}^{t}\tau^{-1-q}(1-\tau)^{q-2}\,d\tau)] \\
& +\frac{1}{1-2q}t^{1-q}(1-t)^{q}[t_0^{q-1}(1-t_0)^{-q}(t_0-q)W \\
& +\frac{1}{k_3}(k_1\int_{t_0}^{t}\tau^{q-1}(1-\tau)^{-1-q}\,d\tau \\
& +g(W)\int_{t_0}^{t}\tau^{q+\eta-2}(1-\tau)^{-\eta-q}\,d\tau + k_2\int_{t_0}^{t}\tau^{q-2}(1-\tau)^{-1-q}\,d\tau)]
\end{aligned}$$

$$
\begin{aligned}
y(t) &= \frac{1}{1-2q} t^{q}(1-t)^{1-q} t_0^{-q}(1-t_0)^{q-1}(1-q-t_0) W \\
&+\frac{1}{1-2q}\frac{1}{k_3} t^{q}(1-t)^{1-q} [-k_1 \frac{1}{q-1}\left(\left(\frac{1}{t_0}-1\right)^{q-1}-\left(\frac{1}{t}-1\right)^{q-1}\right) \\
&-g(W)\frac{1}{q-\eta}\left(\left(\frac{1}{t_0}-1\right)^{q-\eta}-\left(\frac{1}{t}-1\right)^{q-\eta}\right) \\
&-k_2\frac{1}{q(q-1)}\left(\left(\frac{1}{t_0}-1\right)^{q-1}\left(\frac{q-1}{t_0}+1\right)-\left(\frac{1}{t}-1\right)^{q-1}\left(\frac{q-1}{t}+1\right)\right)] \\
&+\frac{1}{1-2q} t^{1-q}(1-t)^{q} t_0^{q-1}(1-t_0)^{-q}(t_0-q) W \\
&+\frac{1}{1-2q}\frac{1}{k_3} t^{1-q}(1-t)^{q} [k_1 \frac{1}{q}\left(\left(\frac{1}{t}-1\right)^{-q}-\left(\frac{1}{t_0}-1\right)^{-q}\right) \\
&+g(W)\frac{1}{\eta+q-1}\left(\left(\frac{1}{t}-1\right)^{1-q-\eta}-\left(\frac{1}{t_0}-1\right)^{1-q-\eta}\right) \\
&+k_2\frac{1}{q(q-1)}\left(\left(\frac{1}{t}-1\right)^{-q}\left(\frac{q-t}{t}\right)-\left(\frac{1}{t_0}-1\right)^{-q}\left(\frac{q-t_0}{t_0}\right)\right)]
\end{aligned}
$$

Finally I still need to pin down the value of t_0. I consider the case where $p \to 1$.

For a worker how has accumulated human capital, I know from the surplus equation that

$$
S^{hc}(p,k) = \frac{\phi\overline{\alpha}(p) - rU}{r+\delta} + K_1^k p^{\frac{1}{2}-\frac{1}{2}\theta_{hc}}(1-p)^{\frac{1}{2}+\frac{1}{2}\theta_{hc}},
$$

so

$$
S^{hc}(1,k) = \frac{\phi\alpha^G - rU}{r+\delta}.
$$

Similarly for a worker has not accumulated human capital, I have

$$
\begin{aligned}
S^{nohc}(p,k) &= \frac{\overline{\alpha}(p) - rU}{r+\delta+\gamma} + \frac{1}{2}\frac{1}{r+\delta+\gamma}\zeta_{nohc}^2 p^2 (1-p)^2 S_{pp}^{nohc}(p,k) \\
&+\frac{\gamma}{r+\delta+\gamma} S^{hc}(p,k),
\end{aligned}
$$

so

$$
S^{nohc}(1,k) = \frac{\alpha^G - rU}{r+\delta+\gamma} + \frac{\gamma}{r+\delta+\gamma} S^{hc}(1,k).
$$

Substituting in for $S^{hc}(1,k)$ leads to

$$
S^{nohc}(1,k) = \frac{\alpha^G - rU}{r+\delta+\gamma} + \frac{\gamma}{r+\delta+\gamma}\frac{\phi\alpha^G - rU}{r+\delta}.
$$

Using the notation above implies that

$$\lim_{t\to 1} y(t) = \frac{\alpha^G - rU}{r+\delta+\gamma} + \frac{\gamma}{r+\delta+\gamma}\frac{\phi\alpha^G - rU}{r+\delta}.$$

In other words, the above equation (implicitly) pins down t_0 (i.e. $\underline{p}^{nohc}$).

Summarizing, I have

$$\begin{aligned}
&\lim_{t\to 1}[\frac{1}{1-2q} t^q (1-t)^{1-q} t_0^{-q} (1-t_0)^{q-1} (1-q-t_0) W \\
&+\frac{1}{1-2q}\frac{1}{k_3} t^q (1-t)^{1-q} [-k_1 \frac{1}{q-1}\left(\left(\frac{1-t_0}{t_0}\right)^{q-1} - \left(\frac{1-t}{t}\right)^{q-1}\right) \\
&-g(W)\frac{1}{q-\eta}\left(\left(\frac{1-t_0}{t_0}\right)^{q-\eta} - \left(\frac{1-t}{t}\right)^{q-\eta}\right) \\
&-k_2 \frac{1}{q(q-1)}\left(\frac{q-1+t_0}{t_0}\left(\frac{1}{t_0}-1\right)^{q-1} - \frac{q-1+t}{t}\left(\frac{1-t}{t}\right)^{q-1}\right)] \\
&+\frac{1}{1-2q} t^{1-q} (1-t)^q t_0^{q-1} (1-t_0)^{-q} (t_0 - q) W \\
&+\frac{1}{1-2q}\frac{1}{k_3} t^{1-q} (1-t)^q [k_1 \frac{1}{q}\left(\left(\frac{1-t}{t}\right)^{-q} - \left(\frac{1-t_0}{t_0}\right)^{-q}\right) \\
&+g(W)\frac{1}{\eta+q-1}\left(\left(\frac{1-t}{t}\right)^{1-q-\eta} - \left(\frac{1-t_0}{t_0}\right)^{1-q-\eta}\right) \\
&+k_2 \frac{1}{q(q-1)}\left(\frac{q-t}{t}\left(\frac{1}{t}-1\right)^{-q} - \frac{q-t_0}{t_0}\left(\frac{1}{t_0}-1\right)^{-q}\right)]] \\
= \; & \frac{\alpha^G - rU}{r+\delta+\gamma} + \frac{\gamma}{r+\delta+\gamma}\frac{\phi\alpha^G - rU}{r+\delta},
\end{aligned}$$

where

$$k_3 = \frac{1}{2}\frac{1}{r+\delta+\gamma}\zeta_{nohc}^2$$

$$k_1 = -\frac{\alpha^G - \alpha^B}{r+\delta}\frac{r+\delta+\gamma\phi}{r+\delta+\gamma}$$

$$g(W) = -\frac{\gamma}{r+\delta+\gamma}K_1^k$$

$$\theta_{hc} = \sqrt{\frac{8(r+\delta)}{\zeta_{hc}^2} + 1}$$

$$\eta = \frac{1}{2} - \frac{1}{2}\theta_{hc}$$

$$k_2 = -\frac{1}{r+\delta+\gamma}\left(\alpha^B - rU + \frac{\gamma\left(\phi\alpha^B - rU\right)}{r+\delta}\right)$$

$$q = \frac{1}{2}\left(1 - \sqrt{\frac{4+k_3}{k_3}}\right).$$

Note that the above equation pins down $\underline{p}^{nohc}(k)$ both for the case where $k > 0$ and also $k = 0$. Moreover that W changes depending on the case considered. In particular for $k = 0$

$$W = 0,$$

whereas when $k = 1$

$$W = E_{p_0} S^{nohc}(p_0, 0).$$

The expression $S^{nohc}(p_0, 0)$ has been solved above (since $y(t) = S^{nohc}(p, k)$).

1.3 On-the-Job Search Derivations

In the model with on-the-job search, when worker moves to another firm, his wage is given by the Nash bargaining. I show here that this is the equilibrium of an auction between the incumbent firm and the firm the worker has contacted (poacher).

The structure of the auction, which closely resembles the one is Moscarini (2005) and is essentially identical to Papageorgiou (2014) with minor modifications, is the following:

When an employed worker meets a new firm there is competition for the worker's services. The competition determines the firm (incumbent or poacher) where the worker becomes employed, as well as a lump-sum transfer from the winning firm to the worker. After the competition is over, the winning firm engages in continuous renegotiation of wages with the worker.

Firm competition is according to the following protocol:

First each firm needs to decide whether to pay $\varepsilon > 0$ and enter the auction. The incumbent decides first and the poacher observes the incumbent's choice and decides whether to enter. Following the poacher's choice, the incumbent has another chance to enter the auction if he hadn't entered the first time.

Second, the auction takes place. If no firm enters the auction, then the worker remains with the incumbent firm and there is no transfer. If one firm enters the auction, then the worker becomes employed at that firm and there is no transfer. If both firms enter the auction, then the transfer is determined by an ascending bid auction and the worker becomes employed at the winning firm. In case of a tie, he remains employed at the incumbent firm.

I consider the subgame perfect Nash equilibrium of this game as $\varepsilon > 0$ goes to zero. As shown in Papageorgiou (2014), in the unique equilibrium, a worker who has contacted another firm, always ends up employed in the firm where the surplus is the highest, there is no lump-sum transfer and his wage is given by (1). The intuition is straightforward. Consider the case where the surplus is higher if the worker moves to the poacher. In that case the incumbent knows that if he enters the auction, he will be outbidded by the poacher. By backward induction, it is optimal for the incumbent to decline entry and avoid paying the entry cost. Similarly in the case where the surplus is higher if the worker stays with the incumbent, the poacher understands that he will be outbid should he enter the auction and chooses not to enter.

The solution to the Nash bargaining problem results in the linear sharing rule

$$\beta J(p,k) = (1-\beta)(W(p,k) - U), \tag{16}$$

which provides the necessary condition to determine the worker's wage.

Shimer (2006)has shown that with on-the-job search the linear sharing rule may not always be bilaterally efficient. In particular, he has argued that if the value of the worker is higher in other firms, the incumbent employer might have an incentive to pay the worker a higher wage in exchange for not searching on the job. For instance, assume the wage prescribed by the Nash bargaining solution in several other firms is only slightly higher than the worker's current wage. Then the current firm may find it profitable to increase the worker's wage so that he does not find it profitable to look for a job in other firms. In other words, the trade-off faced by the incumbent firm is between a slight reduction in profits, but a discrete jump in the expected duration of its match with the worker, which might make the wage increase optimal. The worker's value would also go up following the wage increase. This would in turn imply that the set of feasible payoffs is non-convex, thus violating one of Nash's axioms.

In a framework like the present one, with costless and unobserved job search, such a strategy by the current employer would not work. As Moscarini (2005) notes, if the current employer did offer a higher wage, the worker would have incentive to continue searching on-the-job. If he did contact a firm with higher surplus, the poaching firm can always outbid the incumbent firm in the ensuing auction and offer the worker an (even) higher value. Put differently, changing the current wage does not affect turnover and therefore the duration of the match in the environment studied. Note that this depends on the assumption of costless and unobserved on-the-job search, which rules out this strategy for the firm and preserves the convexity of the set of feasible payoffs. Thus in the present setup, the linear sharing rule (16) is bilaterally efficient.

2 Within Firm Occupational Reallocation

Occupation i	Occupation j	$Flow\,(i,j) + Flow\,(j,i)$	$\frac{\lvert Flow(i,j) - Flow(j,i)\rvert}{Flow(i,j)+Flow(j,i)}$
A	D	968	0.141
A	C	541	0.117
C	D	330	0.055
A	G	300	0.193
A	I	274	0.183
I	J	262	0.107
A	B	219	0.26
J	L	167	0.054
I	L	164	0.098
D	G	161	0.044
C	G	136	0.132
C	L	133	0.173
D	I	119	0.025
C	I	103	0.01
D	J	102	0.235
D	L	101	0.168
G	L	91	0.033
B	D	90	0.067
K	L	89	0.258
E	G	88	0.273
G	I	86	0.07
A	J	79	0.089
I	K	78	0.154
D	K	73	0.151
G	J	64	0.031
A	K	62	0.097
B	I	60	0.067

Table 1: Within Firm Occupational Reallocation. 1996 Panel of Survey of Income and Program Participation. 4-month transitions. Occupational groups defined as A. Managerial and Professional Specialty Occupations (003-199); B. Technical Support (203-235); C. Sales (243-285); D. Administrative Support (303-389); E. Private Household Occupations (403-407); F. Protective Service Occupations (413-427); G. Service (433-469); H. Farming (473-499); I. Precision Production (503-699); J. Machine Operators (703-799); K. Transportation (803-859); L. Handlers (864-889)

Occupation i	Occupation j	$Flow\,(i,j) + Flow\,(j,i)$	$\frac{\lvert Flow(i,j) - Flow(j,i)\rvert}{Flow(i,j)+Flow(j,i)}$
A	L	58	0.172
J	K	54	0.148
A	F	51	0.02
G	G	48	0.25
B	G	45	0.111
H	L	45	0.244
H	I	38	0.211
C	K	36	0.056
D	F	36	0
C	J	35	0.2
B	J	32	0
A	H	31	0.097
G	K	28	0.071
F	G	27	0.111
H	K	26	0.154
B	C	23	0.130
H	J	21	0.429
C	H	18	0.222
D	H	18	0
C	E	16	0.875
A	E	15	0.6
D	E	13	0.692
B	L	11	0.091
C	F	11	0.273
F	I	10	0
E	H	8	0
E	L	8	0.5

Table 2: Within Firm Occupational Reallocation (continued). 1996 Panel of Survey of Income and Program Participation. 4-month transitions. Occupational groups defined as A. Managerial and Professional Specialty Occupations (003-199); B. Technical Support (203-235); C. Sales (243-285); D. Administrative Support (303-389); E. Private Household Occupations (403-407); F. Protective Service Occupations (413-427); G. Service (433-469); H. Farming (473-499); I. Precision Production (503-699); J. Machine Operators (703-799); K. Transportation (803-859); L. Handlers (864-889)

Occupation i	Occupation j	$Flow\,(i,j) + Flow\,(j,i)$	$\frac{\lvert Flow(i,j)-Flow(j,i)\rvert}{Flow(i,j)+Flow(j,i)}$
F	K	7	0.429
E	J	6	0.333
F	J	6	0.667
E	I	5	0.6
B	F	4	0.5
B	K	4	1
F	L	4	0.5
F	H	3	1
B	E	1	1
B	H	1	1

Table 3: Within Firm Occupational Reallocation (continued). 1996 Panel of Survey of Income and Program Participation. 4-month transitions. Occupational groups defined as A. Managerial and Professional Specialty Occupations (003-199); B. Technical Support (203-235); C. Sales (243-285); D. Administrative Support (303-389); E. Private Household Occupations (403-407); F. Protective Service Occupations (413-427); G. Service (433-469); H. Farming (473-499); I. Precision Production (503-699); J. Machine Operators (703-799); K. Transportation (803-859); L. Handlers (864-889)

3 Hierarchical Model of Ability

In this section I solve the model for the case where workers are hierarchically ranked, so that abilities are perfectly correlated across tasks and workers learn about their unobserved productivity. In order to accomplish this I extend the model introduced in Papageorgiou (2014), to the case where firms have more than one occupations. The interested reader can also look the hierarchical setup of Groes et al. (2015).[2]

3.1 The Economy

Time is continuous. There is a population of workers of mass one and a fixed mass of firms. Workers die exogenously at Poisson rate γ and new workers enter the population at the same rate. The discount rate is r for both workers and firms.

There are two types of workers, a high type, H and a low type, L. A worker's type is unknown and drawn at birth. A worker's type is high, H, with probability p_0 which is common knowledge. p_0 is distributed across the population of workers according to a beta distribution with parameters ψ_1 and ψ_2.

A worker can be either employed or unemployed. While unemployed they earn b_u. Workers are born unemployed.

[2] In both papers, a firm has only one occupation and a worker needs to switch firms in order to switch occupations.

Moreover there are two occupations, occupation 1 and occupation 2. The flow output for a worker of type τ in occupation i at time t is given by:

$$dY_t^{\tau i} = a_i^{\tau} dt + \sigma_i dZ_t^{\tau i}$$

where $Z_t^{\tau i}$ is a Wiener process, a_i^{τ} is the occupation and type specific mean, whereas σ_i is the occupation-specific output noise. Output realizations are common knowledge.

I am interested in the hierarchical ability case, where H type workers are better than L type workers in both occupation, i.e.:

$$\alpha_i^H > \alpha_i^L$$

for $i \in \{1, 2\}$. Moreover, without loss of generality, assume that a H type worker performs better in occupation 1:

$$\alpha_1^H > a_2^H$$

To exclude the case where both worker types prefer to be employed in the same occupation, assume that a L type worker performs better in occupation 2:

$$\alpha_2^L > \alpha_1^L$$

I also introduce search frictions. A unemployed worker meets a firm at some Poisson rate λ. Search is costless and an employed worker can also meet firms at some rate $\eta\lambda$, where $\eta \in [0, 1]$. A worker-firm match is destroyed either exogenously at some Poisson rate δ or endogenously. A worker and a firm split the rents generates by search frictions according to Nash bargaining where β denotes the worker's bargaining power. When an employed worker meets another firm, he moves to the firm where his value is the highest after receiving the wage resulting from Nash bargaining.

I examine two cases. The first case is the baseline model where each firm has only one occupation. In this case, an unemployed worker must choose the occupation he searches for employment. The second case is when firms have both occupations available.

3.2 Equilibrium

In this section I examine equilibrium behavior in each of the two cases. In both cases, a worker observes output realizations and updates his posterior belief, p, that he is a high type according to:

$$dp_t = p_t (1 - p_t) \xi_i \frac{dY_t^{\tau i} - \left(p_t a_i^H + (1 - p_t) a_i^L\right) dt}{\sigma_i}$$

where $\xi_i \equiv \frac{a_i^H - a_i^L}{\sigma_i}$.

I next examine each case separately.

In the first case where each firm has only one occupation, worker behavior is identical to that in Papageorgiou (2014). In that setup there exist three unique thresholds, $\underline{p} < \widehat{p} < \overline{p}$, that characterize optimal worker behavior.

In particular, a worker who is unemployed searches for a firm in occupation 1 if his belief that he's a high type, exceeds $\widehat{p}$. An employed worker in occupation 1 optimally quits his firm and becomes unemployed if his posterior reaches $\underline{p}$. Similarly an employed worker in occupation 2 optimally quits his firm if his posterior reaches $\overline{p}$. Finally, an employed worker in occupation 1, searches on-the-job for an occupation 2 firm if $p \in (\underline{p}, \widehat{p}]$, while an employed worker in occupation 2, searches on-the-job for an occupation 1 firm $p \in (\widehat{p}, \overline{p})$.

I now turn to the case where firms have both occupations.

Consider first an unemployed worker. Since now all firms are identical, there's no search decision for the worker: once he meets a firm, he optimally choose which occupation to work in.

Therefore the flow value of an unemployed worker with posterior p is given by:

$$(r + \gamma) U (p) = b_u + \lambda \left(\max_i W_i (p) - U (p) \right) \tag{17}$$

where r is the discount rate, b_u is the unemployment benefit, $W_i (p)$ is the value of a worker employed occupation i, λ is the job finding rate and γ is the death rate.

The Hamilton-Jacobi-Bellman equation for an employed worker in occupation i is given by:

$$(r + \gamma) W_i (p) = w_i (p) + \frac{1}{2} \xi_i^2 p^2 (1 - p)^2 W_i'' (p) - \delta (W_i (p) - U (p)) \tag{18}$$

where $w_i (p)$ is the occupation-specific wage.

Similarly, the flow value of a firm employing the worker is given by:

$$(r+\gamma)\,J_i(p) = \overline{\alpha}_i(p) - w_i(p) + \frac{1}{2}\xi_i^2 p^2 (1-p)^2 J_i''(p) - \delta J_i(p) \tag{19}$$

where $\overline{\alpha}_i(p) = p\alpha_i^H + (1-p)\,\alpha_i^L$.

Finally, the solution to the Nash bargaining implies that:

$$\beta J_i(p) = (1-\beta)\,(W_i(p) - U(p)) \tag{20}$$

In this setup, a worker has no incentive to endogenously quit to unemployment. Given that he can costlessly switch occupations within the firm, his only decision is when to switch occupations. Optimal occupational choice requires that the following three conditions are satisfied:

$$W_1(\widehat{p}) = W_2(\widehat{p}) \tag{21}$$

$$W_1'(\widehat{p}) = W_2'(\widehat{p}) \tag{22}$$

$$W_1''(\widehat{p}) = W_2''(\widehat{p}) \tag{23}$$

The first two conditions above are the standard value matching and smooth-pasting conditions, which concern the level and first derivative of the value functions. The third required condition, which states that the second derivatives of the value functions be equal. The necessity of this last condition is proven in Eeckhout and Weng (2015).

I proceed to solve out for $\widehat{p}$, as well as the value functions and the wage function.

Straightforward derivations imply that the worker's wage is given by:[3]

$$\begin{aligned} w_i(p) &= \beta\overline{a}_i(p) + (1-\beta)\,b_u + \beta\lambda J_i(p) - \frac{1}{2}(1-\beta)\,\xi_i^2 p^2 (1-p)^2 W_i''(p) \\ &\quad + \frac{1}{2}\beta\xi_i^2 p^2 (1-p)^2 J_i''(p) \end{aligned}$$

Using the surplus sharing condition (eq. (20)) and the value of an unemployed worker (eq. (17)), I

[3] In order to obtain this expression, we subtract the worker's flow value of being unemployed (equation (17)) from his flow value of being employed (eq. (18)) and multiply through by $(1-\beta)$. We similarly multiply the flow asset value of a filled vacancy (eq. (19)) by β. We then subtract the above two equations and using the surplus sharing condition (eq. (20)), one obtains the desired expression.

can substitute out for $W_i''(p)$ in the above equation and obtain the wage as a function of $J_i(\cdot)$ only:

$$w_i(p) = \beta\overline{a}_i(p) + (1-\beta)\,b_u + \beta\lambda J_i(p) - \frac{\beta\lambda}{2(r+\gamma)}\xi_i^2 p^2 (1-p)^2 J_i''(p)$$

Replacing the wage into the value of the firm value function (eq. (19)) produces a differential equation with respect to $J_i(\cdot)$:

$$(r+\gamma+\delta+\beta\lambda)\,J_i(p) = (1-\beta)\left(\overline{a}_i(p) - b_u\right) + \frac{r+\gamma+\beta\lambda}{2(r+\gamma)}\xi_i^2 p^2 (1-p)^2 J_i''(p)$$

The general solution to the above differential equation is:

$$\begin{aligned} J_i(p) &= \frac{(1-\beta)\left(\overline{a}_i(p)-b_u\right)}{r+\gamma+\delta+\beta\lambda} + K_i p^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4+h_i}{h_i}}}(1-p)^{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{4+h_i}{h_i}}} \\ &\quad + R_i p^{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{4+h_i}{h_i}}}(1-p)^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4+h_i}{h_i}}} \end{aligned}$$

where $h_i = \frac{1}{2}\frac{r+\gamma+\beta\lambda}{(r+\gamma+\delta+\beta\lambda)(r+\gamma)}\xi_i^2$ and K_i and R_i are undetermined coefficients. For the case of $i=1$, when $p\to 1$, $\lim_{p\to1} R_1 p^{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{4+h_1}{h_1}}}(1-p)^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4+h_1}{h_1}}} = R_1\cdot 1\cdot \lim_{p\to1}(1-p)^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4+h_1}{h_1}}} = +\infty$ which follows from $h_1>0$, and therefore $\sqrt{\frac{4+h_1}{h_1}}>1$ and $\frac{1}{2}\left(1-\sqrt{\frac{4+h_1}{h_1}}\right)<0$. However since the profits of the firm are bounded from above by the total value of the surplus when the worker is known to be a high type, which is finite, it must be the case that $R_1=0$. A similar argument for $i=2$ and $p\to 0$ leads to $K_2=0$.

I need to pin down the values of 3 unknowns: $\widehat{p}$, K_1 and R_2. In order to do so, I use the three conditions mentioned before, equations (21) through (23). Straightforward derivations show that these three conditions lead to:

$$\begin{aligned} &\frac{1-\beta}{r+\gamma+\delta+\beta\lambda}\left(\left(\alpha_1^H-\alpha_1^L-\alpha_2^H+\alpha_2^L\right)\widehat{p}+\alpha_1^L-\alpha_2^L\right) \\ &-\frac{(1-\beta)\left(\alpha_1^H-\alpha_1^L-\alpha_2^H+\alpha_2^L\right)}{r+\gamma+\delta+\beta\lambda} \\ &\widehat{p}(1-\widehat{p})\left(\left(\frac{1}{2}-\widehat{p}\right)\left(1-\left(\frac{\xi_1}{\xi_2}\right)^2\right)+\frac{1}{2}\left(1+\left(\frac{\xi_1}{\xi_2}\right)^2\right)\sqrt{\frac{4+h_2}{h_2}}\right)^{-1}\left(1-\left(\frac{\xi_1}{\xi_2}\right)^2\right) \\ = \;& 0 \end{aligned}$$

r	0.004	γ	0.0033
β	0.3	δ	0.008
b_u	0.6	λ	0.36
ψ_1	4.5	ψ_2	6

Table 4: Fixed Simulation Parameters

$$R_2 = \frac{(1-\beta)\left(\alpha_1^H - \alpha_1^L - \alpha_2^H + \alpha_2^L\right)}{r+\gamma+\delta+\beta\lambda}$$
$$\widehat{p}^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4+h_2}{h_2}}}\left(1-\widehat{p}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{4+h_2}{h_2}}}\left(\left(\frac{1}{2}-\widehat{p}\right)\left(1-\left(\frac{\xi_1}{\xi_2}\right)^2\right)+\frac{1}{2}\left(1+\left(\frac{\xi_1}{\xi_2}\right)^2\right)\sqrt{\frac{4+h_2}{h_2}}\right)^{-1}$$

$$K_1 = R_2\left(\frac{\xi_1}{\xi_2}\right)^2\left(\frac{\widehat{p}}{1-\widehat{p}}\right)^{\sqrt{\frac{4+h_2}{h_2}}}$$

3.3 Model Implications

I now turn to the implications of the two models regarding separation probabilities and wages.

I first begin with the implications regarding the separation rates. In the case where firms only have one occupation, workers may end up separating from their firm for two reasons: a) exogenously, when hit by a δ shock, or b) endogenously either directly to another firm through on-the-job search, or by quitting to unemployment if their posterior hits one of the two separation triggers.

In the case where firms have both occupations, as discussed in the previous section there are no endogenous separations. Since both occupations are available in the firm, a worker need not look for another firm in order to move to the other occupation. As a result separations are only the result of exogenous separations which are the same in both cases. Therefore the following proposition holds:

Proposition 1. *The separation rate is lower in the case where firms have both occupations available.*

Moreover, if one considers a worker in the first case, conditional on his wage, he's still more likely to separate compared to the same worker in a two-occupation firm. The reason is that in the first case, unless his posterior is equal to zero or one, there's always a positive probability that his posterior hits the separation threshold, whereas in the second case the probability of endogenous separation is zero.

Proposition 2. *Conditional on the wage, the separation rate is lower in the case where firms have both occupations available*

a_1^H	a_1^L	a_2^H	a_2^L	ξ_1	ξ_2	η	$\frac{\text{meanwage2}}{\text{meanwage1}}$
1.6	0.3	1.2	0.7	0.3	0.3	0.3	1.035
1.6	0.3	1.2	0.6	0.3	0.3	0.3	1.051
1.6	0.3	1.2	0.7	0.2	0.2	0.3	1.025
1.6	0.3	1.2	0.7	0.45	0.45	0.3	1.032
1.6	0.5	1.2	0.7	0.3	0.3	0.3	1.027
1.6	0.3	1.2	0.8	0.3	0.3	0.1	1.024
1.5	0.3	1.2	0.7	0.3	0.3	0.1	1.017
1.6	0.3	1.2	0.7	0.3	0.3	0.1	1.021
1.5	0.2	1.2	0.7	0.3	0.3	0.1	1.018
1.6	0.2	1.1	0.7	0.3	0.3	0.1	1.016
1.6	0.3	1.3	0.7	0.3	0.3	0.1	1.017
1.6	0.3	1.2	0.7	0.5	0.5	0.1	1.020
1.7	0.3	1.2	0.7	0.3	0.3	0.1	1.023

Table 5: Simulation Parameters

I now turn to the implications regarding wages and show that workers in firms where both occupations are available are better matched and therefore earn higher wages.

In the first case where firms only have one occupation, a worker may prefer to work in a different occupation than his current one, but because of search frictions he stays mismatched. More specifically, these are the workers in occupation 1 with $p \in (\underline{p}, \widehat{p}]$ and those employed in occupation 2 with $p \in (\widehat{p}, \overline{p})$.

In the second case where firms have both occupations, workers who find themselves better matched in the other occupation, can simply switch without any cost: there's no need to remain "mismatched" in the wrong occupation.

I confirm the above intuition by simulating the two cases. Each simulation consisted of 2,000 workers for each case. The parameters in Table 4 are fixed in all simulations. I try out different combinations of the remaining parameters. The results are shown in Table 5: in the last column the mean wage of workers in two occupation firms (case 2) is larger than that of workers in single occupation firms. This is true in every parameter combination, as expected.

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